Exercises for the course “**Mathematical strategies for complex stochastic dynamics**”, Summer 2025, FUB

1. Consider the ODE in $\mathbb{R}^d$

$$\begin{aligned} \frac{dx(t)}{dt} =& f(x(t))\,, \quad t \in [0,T]\,, \\ x(0) =& x, \quad x \in \mathbb{R}^d\,. \end{aligned} \tag{1}$$

Assume that $x(0)$ is a random variable whose probability density is $p_0(x)$. Let $p(x,t)$ be the probability density of $x(t)$ at time $t \in [0,T]$.

(a) Show that

$$\frac{d}{dt} p(x(t),t) = -\text{div} f(x(t))\, p(x(t),t), \quad t \in [0,T]\,.$$

(b) Show that

$$p(x(t),t) = p_0(x) \mathrm{e}^{-\int_0^t (\text{div} f)(x(s))ds}.$$

Hint: Recall that $p(x,t)$ solves the continuity equation

$$\begin{aligned} &\frac{\partial p}{\partial t} + \text{div}(fp) = 0\,, \quad \forall\, t \in (0,T] \\ &p(x,0) = p_0\,. \end{aligned}$$

2. Prove the identity

$$\int_0^t B_s^2 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s ds \tag{2}$$

Hint: Apply Ito’s formula to B_t^3 and then use it to show that both sides satisfy the same SDE (and have the same value zero at $t = 0$).

3. Let B_t be a one-dimensional Brownian motion. Consider the SDE for $X_t = (X_t^1, X_t^2) \in \mathbb{R}^2$:

$$\begin{cases} dX_t^1 = -\frac{1}{2} X_t^1 dt - X_t^2 dB_t \\ dX_t^2 = -\frac{1}{2} X_t^2 dt + X_t^1 dB_t \end{cases} \tag{3}$$

(a) Show that $|X_t|^2$ is constant, i.e. $|X_t| = |X_0|$ for all $t \geq 0$.

(b) Show that $X_t = (\cos(B_t), \sin(B_t))$ solves (3).

Hint: Apply Ito's formula.

4. Consider the Brownian dynamics

$$dX_t = -\nabla V(X_t)\, dt + \sqrt{2\beta^{-1}} dB_t\,, \quad t > 0 \tag{4}$$

where $\beta > 0$ is a constant.

(a) Write down the expression of the infinitesimal generator $\mathcal{L}$ of (4).

(b) Let $\pi(x) = \frac{1}{Z}\mathrm{e}^{-\beta V(x)}$ be the density of Boltzmann distribution, where Z is a normalizing constant. Show that π is invariant under the dynamics (4).

(c) Show that $\mathcal{L}$ is a self-adjoint operator with respect to the weighted inner product $\langle g_1, g_2\rangle_\pi = \int_{\mathbb{R}^d} g_1 g_2 \pi dx$.

Hint: See the lecture note.

5. Consider the Ornstein-Unlenbeck (OU) process

$$dX_t = -\kappa X_t\, dt + \sqrt{2\beta^{-1}} dB_t\,, \quad t > 0\,, \tag{5}$$

where $\kappa, \beta > 0$, and X_0 is fixed.

(a) Show that

$$X_t = \mathrm{e}^{-\kappa t} X_0 + \sqrt{2\beta^{-1}} \int_0^t \mathrm{e}^{-\kappa(t-s)} dB_s, \quad t > 0\,. \tag{6}$$

(b) Compute the mean and variance of X_t.

Hint: Apply Ito's formula to $\mathrm{e}^{-\kappa t} X_t$ and then integrate to get (6). Use (6) and properties of Ito's integral (Ito isometry) to compute the mean and variance.