Lecture 2: SDEs

2025–04–30

1.3 Introduction to SDEs

1.3.1 Brownian motions

Definition 1. A Brownian motion in $\mathbb{R}^d$ is a continuous-in-time stationary Markov process $(B(s))_{s\geq 0}$ with the following properties: for all $t > s \geq 0$,

1. $B(0) = 0$.
2. $B(t)$ is almost surely continuous.
3. $B(t)$ has independent increments, i.e. $B(t) - B(s)$ is independent of $B(s)$.
4. $B(t) - B(s) \sim \mathcal{N}(0, (t-s)I_d)$.

The transition density is

$$p(x,t|y,s) = (2\pi(t-s))^{-\frac{d}{2}} \mathrm{e}^{-\frac{|x-y|^2}{2(t-s)}} \,. \tag{1.1}$$

In particular, the probability density of $B(t)$ at time t is

$$p(x,t) = (2\pi t)^{-\frac{d}{2}} \mathrm{e}^{-\frac{|x|^2}{2t}} \,. \tag{1.2}$$

The following result shows that p in (1.2) solves the heat equation.

Proposition 1. The probability density p satisfies

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta p \,. \tag{1.3}$$

Proof. Taking derivatives in (1.2), we can compute

$$\begin{aligned}\frac{\partial p}{\partial t} &= -\frac{d}{2}(2\pi)(2\pi t)^{-\frac{d}{2}-1}\mathrm{e}^{-\frac{|x|^2}{2t}} + \frac{|x|^2}{2t^2}(2\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|x|^2}{2t}} \\ &= \Big(-\frac{d}{2t} + \frac{|x|^2}{2t^2}\Big)(2\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|x|^2}{2t}} \\ &= \Big(-\frac{d}{2t} + \frac{|x|^2}{2t^2}\Big) p \,,\end{aligned} \tag{1.4}$$

and

$$\frac{\partial p}{\partial x_i} = -\frac{x_i}{t}(2\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|x|^2}{2t}} = -\frac{x_i}{t} p \,. \tag{1.5}$$

Taking derivative with respect to x in (1.5), we obtain

$$\begin{aligned}\Delta p &= \sum_{i=1}^{d} \frac{\partial^2 p}{\partial x_i^2} \\ &= \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\Big(-\frac{x_i}{t}p\Big) \\ &= \sum_{i=1}^{d}\Big(-\frac{p}{t} - \frac{x_i}{t}\frac{\partial p}{\partial x_i}\Big) = \sum_{i=1}^{d}\Big(-\frac{p}{t} + \frac{x_i^2}{t^2}p\Big) = \Big(-\frac{d}{t} + \frac{|x|^2}{t^2}\Big)p\,.\end{aligned} \tag{1.6}$$

The proof is concluded after combining (1.4) and (1.6). □

1.3.2 General SDEs

We start with Ito integration in $\mathbb{R}$.

Let $T > 0$, B_s be a Brownian motion in $\mathbb{R}$, and $f : [0, T] \to \mathbb{R}$ be either deterministic or random (such that $f(t)$ is determined once $B(t)$ is known).

For any integer N, consider a uniform partition of $[0, T]$:

$$0 = t_0 < t_1 < \cdots < t_N = T, \text{ where } \quad t_n = nh, \quad h = \frac{T}{N}\,. \tag{1.7}$$

The Ito integral can be informally defined as:

Definition 2 (Ito integration).

$$\int_0^T f(t)dB_t = \lim_{N\to+\infty} \sum_{n=0}^{N-1} f(t_n)\Delta_n, \text{ where } \Delta_n = B_{t_{n+1}} - B_{t_n}\,. \tag{1.8}$$

Remark. Recall and compare to the definition of the integral

$$\int_0^T f(t)dt = \lim_{N\to+\infty} \sum_{n=0}^{N-1} f(t_n)h\,. \tag{1.9}$$

Remark. $B(t)$ is a Brownian motion implies that

$$\Delta_n = B_{t_{n+1}} - B_{t_n} \sim \mathcal{N}(0, hI_d). \tag{1.10}$$

Therefore, $\mathbb{E}(\Delta_n|B_{t_n}) = 0$ and $\mathbb{E}(\Delta_n^2|B_{t_n}) = h$. Using this fact, we can compute

$$\mathbb{E}\Big(\sum_{n=0}^{N-1} f(t_n)\Delta_n\Big) = \sum_{n=0}^{N-1}\mathbb{E}\big(f(t_n)\Delta_n\big) = \sum_{n=0}^{N-1}\mathbb{E}\big[f(t_n)\mathbb{E}(\Delta_n|B_{t_n})\big] = 0\,, \tag{1.11}$$

and similarly,

$$\mathbb{E}\Big(\sum_{n=0}^{N-1} f(t_n)\Delta_n\Big)^2 = \mathbb{E}\Big[\sum_{n,m=0}^{N-1} f(t_m)f(t_n)\Delta_n\Delta_m\Big] = \mathbb{E}\Big[\sum_{n=0}^{N-1} f^2(t_n)h\Big]. \tag{1.12}$$

Taking the limit $N \to +\infty$, we obtain the identities

$$\begin{aligned}\mathbb{E}\Big(\int_0^T f(t)dB_t\Big) &= 0,\\ \mathbb{E}\Big(\int_0^T f(t)dB_t\Big)^2 &= \int_0^T f^2(t)dt\,.\end{aligned} \tag{1.13}$$

In particular, an infinitesimal version of the second identity is

$$(dB_t)^2 = dt\,,\ \text{ or}\,,\ dB_t dB_s = \delta(t-s)dt\,. \tag{1.14}$$

Now, we give the definition of SDEs.

Let f, σ be two continuous function on $\mathbb{R}$. A Markovian process X_t solves the SDE

$$dX_t = f(X_t)\,dt + \sigma(X_t)dB_t\,,\quad t > 0 \tag{1.15}$$

if and only if

$$X_t = \int_0^t f(X_s)\,ds + \int_0^t \sigma(X_s)dB_s\,,\quad \forall\, t > 0 \tag{1.16}$$

Given a function $g : \mathbb{R} \to \mathbb{R}$, Ito's lemma allows us to compute $dg(X_t)$.

Lemma 1 (Ito's lemma). Assume that X_t solves (1.15). Then, we have

$$dg(X_t) = \Big[g'(X_t)f(X_t) + \frac{1}{2}g''(X_t)\sigma(X_t)^2\Big]dt + \sigma(X_t)g'(X_t)dB_t\,. \tag{1.17}$$

Instead of proving Lemma 1, we provide the following intuition.

Intuition. The identities in (1.14) imply that dB_t is of order $\sqrt{dt}$. Therefore, when computing the expansion of $g(X_{t+h})$, we need to consider 2nd derivative (in order to collect all terms of order t). Concretely, using the SDE (1.15), we have

$$\begin{aligned}dg(X_t) =& g'(X_t)dX_t + \frac{1}{2}g''(X_t)(dX_t)^2\\ =& g'(X_t)f(X_t)dt + g'(X_t)\sigma(X_t)dB_t + \frac{1}{2}g''(X_t)\sigma(X_t)^2 dB_t^2\\ =& \Big(g'(X_t)f(X_t) + \frac{1}{2}g''(X_t)\sigma(X_t)^2\Big)dt + g'(X_t)\sigma(X_t)dB_t\,.\end{aligned} \tag{1.18}$$

In view of Lemma 1, we define the operator

$$\mathcal{L}g = fg' + \frac{1}{2}\sigma^2 g''\,. \tag{1.19}$$

It is called the infinitesimal generator of (1.15). Ito's lemma can be written as

$$dg(X_t) = \mathcal{L}g(X_t)dt + \sigma(X_t)g'(X_t)dB_t\,. \tag{1.20}$$

The following result is a simple application of Lemma 1.

Proposition 2. For a smooth function $g : \mathbb{R} \to \mathbb{R}$. We have

$$\mathbb{E}\big(g(X_t)\big) = \int_0^t \mathbb{E}\big(\mathcal{L}g(X_s)\big)ds\,, \quad \text{or} \quad \frac{d}{dt}\mathbb{E}\big(g(X_t)\big) = \mathbb{E}\big(\mathcal{L}g(X_t)\big)\,. \tag{1.21}$$

Proof. Applying Ito's formula in (1.22), we have

$$g(X_t) = \int_0^t \mathcal{L}g(X_s)ds + \int_0^t g'(X_s)\sigma(X_s)dB_s\,. \tag{1.22}$$

Taking expectation on both sides above, the Ito's integration vanishes thanks to the first identity in (1.13), and we obtain

$$\mathbb{E}\big(g(X_t)\big) = \int_0^t \mathbb{E}\big(\mathcal{L}g(X_s)\big)ds\,.$$

□

1.3.3 Semigroup and Fokker-Planck equation

Define

$$(T_t g)(x) = \mathbb{E}\big[g(X(t))|X(0) = x\big], \quad \forall t \geq 0\,, \forall g\,. \tag{1.23}$$

Proposition 3. $T_0 = \mathrm{id}$ and, for $t, s \geq 0$, $T_{t+s} = T_t \circ T_s$.

Proof. Using the property of conditional expectation, we derive

$$\begin{aligned}(T_{t+s}g)(x) =& \mathbb{E}\big[g(X(t+s))|X(0) = x\big] \\ =& \mathbb{E}\Big(\mathbb{E}\big[g(X(t+s))|X(t)\big]\Big|X(0) = x\Big) \\ =& \mathbb{E}\Big((T_s g)(X(t))\Big|X(0) = x\Big) \\ =& (T_t \circ T_s g)(x)\,.\end{aligned} \tag{1.24}$$

□

Proposition 4.

$$\frac{d}{dt}T_t g = \mathcal{L}T_t g\,. \tag{1.25}$$

Proof. Assume $X_0 = x$. Proposition 2 implies that

$$\frac{d}{dt}|_{t=0}T_t g(x) = \mathbb{E}(\mathcal{L}g(X_0)) = (\mathcal{L}g)(x)\,.$$

This shows that (1.27) holds at $t = 0$. For $t > 0$, applying Proposition 3, we get

$$\frac{d}{dt}T_t g(x) = \frac{d}{ds}|_{s=0} T_{t+s} g = \frac{d}{ds}|_{s=0} T_s\big(T_t g\big) = \mathcal{L}(T_t g)\,.$$

$\square$

Remark. The result above implies that $\mathcal{L}$ is the infinitesimal generator of the semigroup T_t. We often write $T_t = \mathrm{e}^{t\mathcal{L}}$, for $t \geq 0$.

Now, we introduce Fokker-Planck equation. Let us first introduce the adjoint of $\mathcal{L}$.

Definition 3 (Adjoint operator of $\mathcal{L}$)**.** The adjoint operator of $\mathcal{L}$, denoted by $\mathcal{L}^\top$, is defined by

$$\int_{\mathbb{R}^d} \big[(\mathcal{L}^\top g)(x)\big] g_1(x) dx = \int_{\mathbb{R}^d} g(x)\big[\mathcal{L} g_1(x)\big] dx\,, \tag{1.26}$$

for any two smooth functions g and g_1.

Lemma 2. $\mathcal{L}^\top g = -(fg)' + \frac{1}{2}(\sigma^2 g)''\,.$

Proof. For the right hand side of (1.28), using (1.21) and integration by parts, we have

$$\int_{\mathbb{R}} g(x)\big[\mathcal{L} g_1(x)\big] dx = \int_{\mathbb{R}} g(x)\big(f g_1' + \frac{1}{2}\sigma^2 g_1''\big) dx = \int_{\mathbb{R}} \Big[-(fg)' + \frac{1}{2}(\sigma^2 g)''\Big](x) g_1(x) dx\,. \tag{1.27}$$

Combining (1.28), we have

$$\int_{\mathbb{R}} \big[(\mathcal{L}^\top g)(x)\big] g_1(x) dx = \int_{\mathbb{R}} \Big[-(fg)' + \frac{1}{2}(\sigma^2 g)''\Big](x) g_1(x) dx\,. \tag{1.28}$$

Since the above identity holds for any g_1, we conclude that $\mathcal{L}^\top g = -(fg)' + \frac{1}{2}(\sigma^2 g)''$. $\square$

Let $p(x,t)$ denote the probability density of X_t at time t, starting from $X_s = y$.

Proposition 5. The density $p(x,t)$ satisfies the Fokker-Planck equation

$$\begin{aligned} \frac{\partial p}{\partial t} &= \mathcal{L}^\top p\,, \quad t > s\,, \\ p(x,s) &= \delta(x-y), \end{aligned} \tag{1.29}$$

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be a test function. We compute $\frac{d}{dt}\mathbb{E}(g(X_t))$. On the one hand, since p is the density of X_t at time t, we have

$$\frac{d}{dt}\mathbb{E}(g(X_t)) = \frac{d}{dt}\int_{\mathbb{R}} g(x)p(x,t)dx = \int_{\mathbb{R}} g(x)\frac{\partial p}{\partial t}(x,t)dx\,. \tag{1.30}$$

On the other hand, applying Proposition 2, we get

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\big(g(X_t)\big) &= \mathbb{E}\big(\mathcal{L}g(X_t)\big)\\ &= \int_{\mathbb{R}} \mathcal{L}g(x)p(x,t)dx \\ &= \int_{\mathbb{R}} g(x)\mathcal{L}^\top p(x,t)dx\,,\end{aligned} \tag{1.31}$$

where the last equality follows from the definition of $\mathcal{L}^\top$ in (1.28). Combining (1.32) and (1.33), we obtain

$$\int_{\mathbb{R}} g(x)\frac{\partial p}{\partial t}(x,t)dx = \int_{\mathbb{R}} g(x)\mathcal{L}^\top p(x,t)dx\,. \tag{1.32}$$

Since the equality above holds for a general test function g, we conclude that p solves the equation (1.31). □

The results can be extended to SDEs in $\mathbb{R}^d$. The generator is

$$\mathcal{L}g = f\cdot\nabla g + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^\top)_{ij}\frac{\partial^2 g}{\partial x_i\partial x_j}\,. \tag{1.33}$$

Its adjoint is

$$\mathcal{L}^\top g = -\mathrm{div}(fg) + \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2\big((\sigma\sigma^\top)_{ij}g\big)}{\partial x_i\partial x_j}\,. \tag{1.34}$$

We look at a concrete example.

Example (Brownian motion in $\mathbb{R}^d$)**.** For Brownian motion $X_t = B_t$, the SDE is

$$dX_t = dB_t\,. \tag{1.35}$$

In this case, it corresponds to the general form (1.15) with $f = 0$ and $\sigma = I_d$. From (1.21) and Lemma 2, we obtain the generator $\mathcal{L}$ and its adjoints

$$\mathcal{L} = \mathcal{L}^\top = \frac{1}{2}\Delta. \tag{1.36}$$

Therefore, the Fokker-Planck equation in Proposition 5 reduces to the heat equation in Proposition 1.