Lecture 12: Asymptotic analysis

2025–07–16

In this lecture, we consider problems that involve a small parameter $\varepsilon > 0$. We study the limiting behavior of such problems as $\varepsilon \to 0$ using asymptotic analysis techniques. For a systemetic introduction of asymptotic analysis techniques for stochastic dynamics, see [1] and [2].

1.1 Invariant manifolds for ODEs

Consider the ordinary equation (ODE)

$$\begin{aligned} \frac{dx_t}{dt} =& f(x_t, y_t)\,, \\ \frac{dy_t}{dt} =& \frac{1}{\varepsilon} g(x_t, y_t)\,, \end{aligned} \tag{1.1}$$

where $(x_t, y_t) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$, $f : \mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^k$, $g : \mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$, and $\varepsilon > 0$ is a small parameter. Our goal is to study the dynamics (1.1) as $\varepsilon \to 0$.

Intuitively, when ε is small, the dynamics of y_t is much faster than the dynamics of x_t because of the factor $1/\varepsilon$ in the equation of y_t. Let $\varphi_x^t(y)$ be the flow map of the ODE

$$\frac{d\varphi_x^t(y)}{dt} = g(x, \varphi_x^t(y)), \quad \forall\ t \geq 0, \quad \text{and } \varphi_x^0(y) = y\,, \tag{1.2}$$

where $x \in \mathbb{R}^k$ is a fixed parameter and $y \in \mathbb{R}^{d-k}$ is the initial state. We assume that, as $t \to +\infty$, the dynamics (1.2) converges to a limiting state that only depends on x, i.e.

$$\lim_{t\to+\infty} \varphi_x^t(y) = \eta(x)\,, \tag{1.3}$$

and the speed of convergence is uniform in x. Roughly speaking, under this assumption, if we consider time t that is much smaller than 1 so that x_t remains close to x_0, the state y_t satisfies $y_t \approx \varphi_{x_0}^{t/\varepsilon}(y_0) \approx \eta(x_0)$, implying that it quickly reaches equilibrium. If we focus on larger time t that is order $\mathcal{O}(1)$, y_t will behave like a "slave variable" of x_t, such that $y_t \approx \eta(x_t)$. In the following, we justify the above heuristics and derive the limiting equation of x_t.

Asymptotic expansion. We consider the invariant manifold of (1.1) and assume that the invariant manifold can be parametrized as the graph $y = \Phi(x)$, where $\Phi : \mathbb{R}^k \to \mathbb{R}^{d-k}$. Since the manifold is invariant under the dynamics (1.1), the tangent direction of (x_t, y_t) must belong to the tangent space of the invariant manifold, which implies

$$\frac{dy_t}{dt} = \nabla\Phi(x_t)\frac{dx_t}{dt}\,, \tag{1.4}$$

where $\nabla\Phi : \mathbb{R}^k \to \mathbb{R}^{(d-k)\times k}$. Combining (1.4) and (1.1), we know that Φ must satisfy the PDE

$$\frac{1}{\varepsilon} g(x, \Phi(x)) = \nabla\Phi(x) f(x, \Phi(x))\,. \tag{1.5}$$

We seek solutions to (1.5) as a power series

$$\Phi(x) = \Phi_0(x) + \varepsilon\Phi_1(x) + \varepsilon^2\Phi_2(x) + \cdots\,. \tag{1.6}$$

Substituting (1.6) into (1.5), applying Taylor's expansion, and collecting terms of order ε^{-1} and terms of order ε^0, respectively, we obtain the equations of Φ_0 and Φ_1 as

$$\begin{aligned} & g(x, \Phi_0(x)) = 0\,, \\ & \nabla_y g(x, \Phi_0(x))\Phi_1(x) = \nabla\Phi_0(x) f(x, \Phi_0(x))\,. \end{aligned} \tag{1.7}$$

Notice that, since $\eta(x)$ in (1.3) is the limiting state of the ODE flow (1.2), it satisfies $g(x, \eta(x)) = 0$. Therefore, to solve Φ_0, we can choose

$$\Phi_0(x) = \eta(x)\,. \tag{1.8}$$

For Φ_1, assuming that $\nabla_y g(x, \eta(x)) \in \mathbb{R}^{(d-k)\times(d-k)}$ is invertible, then from (1.7) we can obtain

$$\Phi_1(x) = \big(\nabla_y g(x, \eta(x))\big)^{-1} \nabla\eta(x) f(x, \eta(x))\,. \tag{1.9}$$

Substituting (1.8)–(1.9) into (1.6), we obtain

$$\Phi(x) = \eta(x) + \varepsilon\big(\nabla_y g(x, \eta(x))\big)^{-1} \nabla\eta(x) f(x, \eta(x)) + \mathcal{O}(\varepsilon^2)\,. \tag{1.10}$$

Therefore, for the function f in (1.1), applying Taylor's expansion, we obtain

$$\begin{aligned} f(x, \Phi(x)) =& f(x, \eta(x)) + \varepsilon\nabla_y f(x, \eta(x))\big(\nabla_y g(x, \eta(x))\big)^{-1} \nabla\eta(x) f(x, \eta(x)) + \mathcal{O}(\varepsilon^2) \\ =& \Big(I + \varepsilon\nabla_y f(x, \eta(x))\big(\nabla_y g(x, \eta(x))\big)^{-1} \nabla\eta(x)\Big) f(x, \eta(x)) + \mathcal{O}(\varepsilon^2)\,. \end{aligned} \tag{1.11}$$

Limiting equation. Substituting the leading order of the expansion (1.11) into the ODE (1.1), we obtain the approximation of (1.1) in the limit $\varepsilon \to 0$

$$\frac{dX_t}{dt} = F_0(X_t)\,, \tag{1.12}$$

where the vector field $F_0 : \mathbb{R}^k \to \mathbb{R}^k$ is defined as

$$F_0(x) = f(x, \eta(x))\,, \quad x \in \mathbb{R}^k\,.$$

If we keep the term of order ε in (1.11), we obtain the refined limiting ODE

$$\frac{dX_t}{dt} = F_0(X_t) + \varepsilon F_1(X_t)\,, \tag{1.13}$$

where the vector field $F_1 : \mathbb{R}^k \to \mathbb{R}^k$ is defined as

$$F_1(x) = \nabla_y f(x, \eta(x))\big(\nabla_y g(x, \eta(x))\big)^{-1} \nabla\eta(x) f(x, \eta(x))\,, \quad x \in \mathbb{R}^k\,.$$

1.2 Averaging for SDEs

Similar to (1.14), we consider the equation

$$\begin{aligned} \frac{dx_t}{dt} =& f(x_t, y_t)\,, \\ dy_t =& \frac{1}{\varepsilon} g(x_t, y_t) dt + \frac{1}{\sqrt{\varepsilon}} \alpha(x_t, y_t)\, dw_t\,, \end{aligned} \tag{1.14}$$

where w_t is a standard Brownian motion in $\mathbb{R}^{d-k}$, $\alpha : \mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$ is the diffusion coefficient, and functions f and g are defined in the same way as in (1.1). Our goal is again to study the limiting behavior of the process in (1.14) as $\varepsilon \to 0$.

Let $\varphi_x^t(y)$ be the solution to the fast dynamics in (1.14) with fixed x. Precisely, for fixed $x \in \mathbb{R}^k$, $\varphi_x^t(y)$ solves

$$d\varphi_x^t(y) = g(x, \varphi_x^t(y))dt + \alpha(x, \varphi_x^t(y))\, dw_t,\ \forall\, t \geq 0, \quad \text{and } \varphi_x^0(y) = y\,. \tag{1.15}$$

As in the previous section, for time t much smaller than 1, the variable y_t in (1.14) is given by $y_t \approx \varphi_{x_0}^{t/\varepsilon}(y_0)$. If (1.15) is ergodic, then y_t will approach its (x-dependent) invariant density within time t that is large compared to ε. Therefore, intuitively, we can derive a limiting equation for x_t by averaging y in the equation of x_t with respect to the invariant density of (1.15).

The generator of (1.14) can be written as

$$\mathcal{L} = \frac{1}{\varepsilon}\mathcal{L}_0 + \mathcal{L}_1\,, \tag{1.16}$$

where

$$\begin{aligned} \mathcal{L}_0 =& g(x,y)\cdot\nabla_y + \frac{1}{2}\sum_{i,j=1}^{d-k}(\alpha\alpha^\top)(x,y)_{ij}\frac{\partial^2}{\partial y_i y_j}\,,\\ \mathcal{L}_1 =& f(x,y)\cdot\nabla_x\,. \end{aligned} \tag{1.17}$$

Fast dynamics. Assume that x is fixed. Note that $\mathcal{L}_0$ in (1.17) is the generator of the dynamics (1.15). We assume that, for any $x \in \mathbb{R}^k$, the dynamics (1.15) is ergodic and the unique invariant density is $\rho^\infty(y;x)$. From previous lectures, we know that $\rho^\infty(y;x)$ solves the equation

$$\mathcal{L}_0^\top \rho^\infty(y;x) = 0\,, \tag{1.18}$$

where $\mathcal{L}_0^\top$ is the adjoint of $\mathcal{L}_0$ or, in integral form,

$$\int_{\mathbb{R}^{d-k}} \mathcal{L}_0\varphi(y)\rho^\infty(y;x)dy = 0\,, \tag{1.19}$$

for a general function $\varphi : \mathbb{R}^{d-k} \to \mathbb{R}$. From (1.17), it is obvious that $\mathcal{L}_0\varphi \equiv 0$ if φ is a constant function of y. We assume that the converse is also true, namely,

$$\mathcal{L}_0\varphi(y) = 0,\ \forall y \quad\Longrightarrow\quad \varphi(y) \equiv C\,. \tag{1.20}$$

In other words, we assume that $\ker(\mathcal{L}_0)$ is a one-dimensional space spanned by the constant function.

Asymptotic expansion. To study the limiting behavior of the process (1.14), we consider the quantity

$$v(x,y,t) = \mathbb{E}\Big(\phi(x_t)\,\Big|\, x_0 = x, y_0 = y\Big) \tag{1.21}$$

for a function $\phi : \mathbb{R}^k \to \mathbb{R}$. From previous lectures, we know that v solves the PDE

$$\begin{cases} \frac{\partial v}{\partial t} = \mathcal{L}v\,, & t > 0\,,\\ v = \phi, & t = 0\,. \end{cases} \tag{1.22}$$

We want to seek a solution to (1.22) in the form of the expansion

$$v = v_0 + \varepsilon v_1 + \mathcal{O}(\varepsilon^2)\,. \tag{1.23}$$

Substituting (1.23) into (1.22), and using (1.16), we obtain

$$\frac{\partial v_0}{\partial t} = \frac{1}{\varepsilon}\mathcal{L}_0 v_0 + \mathcal{L}_1 v_0 + \mathcal{L}_0 v_1 + \mathcal{O}(\varepsilon)\,. \tag{1.24}$$

Collecting terms of order ε^{-1} and order ε^0, respectively, we obtain

$$\begin{aligned} \mathcal{L}_0 v_0 =& 0\,, \\ \frac{\partial v_0}{\partial t} =& \mathcal{L}_1 v_0 + \mathcal{L}_0 v_1\,. \end{aligned} \tag{1.25}$$

Note that the first equation in (1.25) and the assumption (1.20) imply that v_0 is independent of y, i.e.

$$v_0(x, y, t) = v_0(x, t).$$

From the initial condition in (1.22), we have $v_0(x, 0) = \phi(x)$.

To obtain a closed equation for v_0, we multiply both sides of the second equation in (1.25) by the invariant density $\rho^\infty(y; x)$ and then integrate with respect to y, which gives

$$\int_{\mathbb{R}^{d-k}} \frac{\partial v_0}{\partial t}\rho^\infty(y;x)dy = \int_{\mathbb{R}^{d-k}} \mathcal{L}_1 v_0 \rho^\infty(y;x)dy + \int_{\mathbb{R}^{d-k}} \mathcal{L}_0 v_1 \rho^\infty(y;x)dy\,.$$

Using the fact that v_0 is independent of y, applying the expression of $\mathcal{L}_1$ in (1.17) and the identity in (1.19), we obtain

$$\frac{\partial v_0}{\partial t}(x, t) = F(x) \cdot \nabla_x v_0(x, t) \tag{1.26}$$

for the leading term v_0, where

$$F(x) = \int_{\mathbb{R}^{d-k}} f(x, y)\rho^\infty(y; x)dy\,, \quad x \in \mathbb{R}^k\,. \tag{1.27}$$

Note that the solution to (1.26) can be solved using the method of characteristic as

$$v_0(x, t) = v_0(X_t, 0) = \phi(X_t), \quad t > 0,$$

where X_t is the solution to the ODE

$$\frac{dX_t}{dt} = F(X_t)\,, \quad X_0 = x\,. \tag{1.28}$$

Hence, the expansion $v = v_0 + \mathcal{O}(\varepsilon)$ becomes

$$\mathbb{E}\Big(\phi(x_t)\,\Big|\, x_0 = x, y_0 = y\Big) = \phi(X_t) + \mathcal{O}(\varepsilon)\,. \tag{1.29}$$

The derivation above implies that, as $\varepsilon \to 0$, the limiting dynamics of x_t in (1.14) is given by (1.28).

Example. Consider the SDE

$$\begin{aligned} \frac{dx_t}{dt} =& (1-2y_t^2)x_t\,, \\ dy_t =& -\frac{y_t}{\varepsilon}dt + \sqrt{\frac{2}{\varepsilon}}\,dw_t\,, \end{aligned} \tag{1.30}$$

where $x_t, y_t \in \mathbb{R}$. The invariant density for the fast dynamics (y_t with fixed x_t) is the standard Gaussian

$$\rho^\infty(y) = (2\pi)^{\frac{1}{2}} \mathrm{e}^{-\frac{y^2}{2}}.$$

From (1.27), we obtain

$$F(x) = \Big(\int_{\mathbb{R}} (1-2y^2)\rho^\infty(y)dy\Big)x = -x.$$

Therefore, the limiting ODE of (1.30) is

$$\frac{dX_t}{dt} = -X_t\,.$$

1.3 Mean first exit time

We consider the one-dimensional Brownian dynamics

$$dX_t = -V'(X_t)\,dt + \sqrt{2\varepsilon}dB_t, \quad t \geq 0\,, \tag{1.31}$$

where $V : \mathbb{R} \to \mathbb{R}$ is a potential and $\varepsilon > 0$ is a small parameter.

Assume that $\lim\limits_{|x|\to\infty} V(x) \to +\infty$. Furthermore, x_0 is a local minimal state of V and the basin of attraction of x_0 is

$$D = \Omega(x_0) = (-\infty, x_c), \tag{1.32}$$

where $x_c > x_0$ is a saddle point of the potential V. In particular, (1.32) implies that x_0 is the only local minimal state of V in $(-\infty, x_c)$.

For $x \in \mathbb{R}$, we define the mean first exit time (MFET) from the set D as

$$g(x) = \mathbb{E}(\tau_x)\,, \quad \text{where} \quad \tau_x = \inf\big\{t > 0 \,\big|\, X_t \notin D, X_0 = x\big\}\,.$$

Applying Ito's formula, we can verify that g solves the PDE

$$\begin{aligned} &\mathcal{L}g = -1\,, \quad x \in (-\infty, x_c)\,, \\ &g(x_c) = 0\,, \end{aligned} \tag{1.33}$$

where $\mathcal{L}$ is the generator of (1.31), given by

$$\mathcal{L} = -\frac{dV}{dx}\frac{d}{dx} + \varepsilon\frac{d^2}{dx^2}\,. \tag{1.34}$$

Our goal is to study the solution to (1.33) when $\varepsilon \to 0$.

Intuitively, starting from the vicinity of the local minimal state x_0, the average time for the system to leave the set D grows exponentially as $\varepsilon \to 0$. Based on this observation, we define

$$g = \mathrm{e}^{\frac{K}{\varepsilon}} w \tag{1.35}$$

for some $K > 0$ (whose value will be determine later) such that w is of order $\mathcal{O}(1)$ as $\varepsilon \to 0$. Then, it is easy to verify that w solves

$$\begin{aligned} & -\frac{dV}{dx}\frac{dw}{dx} + \varepsilon \frac{d^2 w}{dx^2} = -\mathrm{e}^{-\frac{K}{\varepsilon}}, \quad x \in (-\infty, x_c)\,, \\ & w(x_c) = 0\,. \end{aligned} \tag{1.36}$$

In order to study the solution to (1.36), we consider asymptotic expansion of w for x close to x_c (boundary of D) and for x away from x_c.

Step 1. First, away from x_c, we consider the expansion

$$w(x) = w_0(x) + \varepsilon w_1(x) + \mathcal{O}(\varepsilon^2)\,. \tag{1.37}$$

Substituting (1.37) into (1.36) and collecting the term of order $\mathcal{O}(1)$, we obtain

$$-\frac{dV}{dx}\frac{dw_0}{dx} = 0\,, \quad x \in (-\infty, x_c)\,,$$

which implies that w_0 is constant under the ODE

$$\frac{dX_t}{dt} = -V'(X_t)\,. \tag{1.38}$$

Since $D = (-\infty, x_c)$ is the basin of attraction associated to x_0, any state $x \in (-\infty, x_c)$ converges to x_0 under the ODE (1.38). Therefore, we conclude that

$$w_0(x) \equiv w_0(x_0) = C_0\,. \tag{1.39}$$

Step 2. Second, near the boundary $x = x_c$, we "zoom in" the boundary layer by considering the change of variables

$$x = x_c - \sqrt{\varepsilon} z\,.$$

Then, we have

$$\begin{aligned} \frac{dw}{dx} &= \frac{dw}{dz}\frac{dz}{dx} = -\frac{1}{\sqrt{\varepsilon}}\frac{dw}{dz}\,, \\ \frac{d^2 w}{dx^2} &= \frac{d^2 w}{dz^2}\Big(\frac{dz}{dx}\Big)^2 = \frac{1}{\varepsilon}\frac{d^2 w}{dz^2}\,. \end{aligned} \tag{1.40}$$

Moreover, note that x_c is a saddle point implies $\frac{dV}{dx}(x_c) = 0$. Therefore, Taylor expansion gives

$$V'(x) = V''(x_c)(x - x_c) + \mathcal{O}(|x - x_c|^2) = \sqrt{\varepsilon}\kappa z + \mathcal{O}(\varepsilon)\,, \tag{1.41}$$

where

$$\kappa = -V''(x_c)\,. \tag{1.42}$$

Substituting (1.41) and (1.40) into (1.36), we obtain

$$\begin{aligned}
&\frac{d^2w}{dz^2}(z) + \kappa z\frac{dw}{dz}(z) + \mathcal{O}(\sqrt{\varepsilon}) = 0, \quad z \in (0, +\infty)\,,\\
&w(0) = 0\,.
\end{aligned} \tag{1.43}$$

To analyze the solution to (1.43), let us consider the expansion

$$w(z) = w_0(z) + \sqrt{\varepsilon}w_1(z) + \mathcal{O}(\varepsilon)\,. \tag{1.44}$$

Substituting (1.44) into (1.43), we obtain the equation for w_0

$$\begin{aligned}
&\frac{d^2w_0}{dz^2}(z) + \kappa z\frac{dw_0}{dz}(z) = 0, \quad z \in (0, +\infty)\,,\\
&w_0(0) = 0\,.
\end{aligned} \tag{1.45}$$

Note that (1.45) can be solved by direct integration, which gives

$$w_0(z) = C_1 \int_0^z \mathrm{e}^{-\frac{\kappa}{2}\eta^2} d\eta\,, \tag{1.46}$$

where C_1 is a constant such that

$$C_1 = \frac{dw_0}{dz}(0). \tag{1.47}$$

Step 3. Sending $z \to +\infty$ in (1.46), and using the identity for Gaussian integral, we obtain

$$\lim_{z\to+\infty} w_0(z) = C_1 \int_0^\infty \mathrm{e}^{-\frac{\kappa}{2}\eta^2} d\eta = C_1\sqrt{\frac{\pi}{2\kappa}}\,. \tag{1.48}$$

On the other hand, as $z \to +\infty$, the limit (1.48) should match the leading term in the previous analysis in (1.39). Therefore, the two constants C_0 and C_1 are related by

$$C_1 = C_0\sqrt{\frac{2\kappa}{\pi}}\,. \tag{1.49}$$

Step 4. To compute C_0 and C_1, we multiply both sides of the differential equation in (1.36) by $\mathrm{e}^{-\frac{1}{\varepsilon}V(x)}$ and integrate over D, which gives

$$\int_{-\infty}^{x_c} \Big(-\frac{dV}{dx}\frac{dw}{dx} + \varepsilon\frac{d^2w}{dx^2}\Big)\mathrm{e}^{-\frac{1}{\varepsilon}V(x)}dx = -\mathrm{e}^{-\frac{K}{\varepsilon}} \int_{-\infty}^{x_c} \mathrm{e}^{-\frac{1}{\varepsilon}V(x)}dx\,. \tag{1.50}$$

For the left hand side of (1.50), we can derive

$$\begin{aligned}
&\int_{-\infty}^{x_c} \Big(-\frac{dV}{dx}\frac{dw}{dx} + \varepsilon\frac{d^2w}{dx^2}\Big)\mathrm{e}^{-\frac{1}{\varepsilon}V(x)}dx\\
=&\varepsilon\int_{-\infty}^{x_c} \frac{d}{dx}\Big(\frac{dw}{dx}\mathrm{e}^{-\frac{1}{\varepsilon}V(x)}\Big)dx\\
=&\varepsilon\Big(\frac{dw}{dx}\mathrm{e}^{-\frac{1}{\varepsilon}V(x)}\Big)\Big|_{-\infty}^{x_c}\\
=&\varepsilon\frac{dw}{dx}(x_c)\mathrm{e}^{-\frac{1}{\varepsilon}V(x_c)}\,,
\end{aligned}$$

where we used the fact that $\lim\limits_{x\to-\infty} \frac{dw}{dx}(x)\mathrm{e}^{-\frac{1}{\varepsilon}V(x)} = 0$, since we have assumed that $\lim\limits_{x\to-\infty} V(x) = +\infty$. Combining the above derivation with (1.40), (1.47) and (1.49), we obtain

$$\begin{aligned}
&\int_{-\infty}^{x_c} \Big(-\frac{dV}{dx}\frac{dw}{dx} + \varepsilon\frac{d^2w}{dx^2}\Big)\mathrm{e}^{-\frac{1}{\varepsilon}V(x)}dx \\
=&\varepsilon\frac{dw}{dx}(x_c)\mathrm{e}^{-\frac{1}{\varepsilon}V(x_c)} \\
=&-\sqrt{\varepsilon}\frac{dw}{dz}(0)\mathrm{e}^{-\frac{1}{\varepsilon}V(x_c)} \\
=&-\sqrt{\varepsilon}\big(C_1 + \mathcal{O}(\sqrt{\varepsilon})\big)\mathrm{e}^{-\frac{1}{\varepsilon}V(x_c)} \\
=&-\sqrt{\varepsilon}\Big(C_0\sqrt{\frac{2\kappa}{\pi}} + \mathcal{O}(\sqrt{\varepsilon})\Big)\mathrm{e}^{-\frac{1}{\varepsilon}V(x_c)}\,.
\end{aligned} \tag{1.51}$$

For the integral on the right hand side of (1.50), noticing that x_0 is the unique local minimal state of $V(x)$ in $(-\infty, x_1)$, applying Laplace's method (see Lemma 1 below), we obtain

$$\int_{-\infty}^{x_c} \mathrm{e}^{-\frac{1}{\varepsilon}V(x)}dx = \sqrt{\varepsilon}\mathrm{e}^{-\frac{1}{\varepsilon}V(x_0)}\Big(\sqrt{\frac{2\pi}{V''(x_0)}} + o(1)\Big)\,. \tag{1.52}$$

Substituting (1.51) and (1.52) into (1.50), we obtain

$$-\sqrt{\varepsilon}\Big(C_0\sqrt{\frac{2\kappa}{\pi}} + \mathcal{O}(\sqrt{\varepsilon})\Big)\mathrm{e}^{-\frac{1}{\varepsilon}V(x_c)} = -\sqrt{\varepsilon}\mathrm{e}^{-\frac{K}{\varepsilon}}\mathrm{e}^{-\frac{1}{\varepsilon}V(x_0)}\Big(\sqrt{\frac{2\pi}{V''(x_0)}} + o(1)\Big)\,, \tag{1.53}$$

from which we can solve (recall that $\kappa = -V''(x_c)$ in (1.42))

$$K = V(x_c) - V(x_0)\,, \quad C_0 = \frac{\pi}{|V''(x_c)|}\sqrt{\frac{|V''(x_c)|}{V''(x_0)}}\,.$$

Therefore, from (1.35), (1.37) and (1.39), we conclude that

$$g(x) = \mathrm{e}^{\frac{K}{\varepsilon}}w(x) = \mathrm{e}^{\frac{V(x_c)-V(x_0)}{\varepsilon}}\Big(\frac{\pi}{|V''(x_c)|}\sqrt{\frac{|V''(x_c)|}{V''(x_0)}} + \mathcal{O}(\varepsilon)\Big)\,.$$

Lemma 1 (Laplace's method)**.** Assume that function $f : [a,b] \to \mathbb{R}$ is C^3 and attains its unique minimum at $x = x_0 \in (a,b)$. Further assume that $f''(x_0) > 0$. Then, as $\varepsilon \to 0$, we have

$$\int_a^b \mathrm{e}^{-\frac{f(x)}{\varepsilon}}dx = \sqrt{\varepsilon}\mathrm{e}^{-\frac{f(x_0)}{\varepsilon}}\Big(\sqrt{\frac{2\pi}{f''(x_0)}} + o(1)\Big)\,.$$

For simplicity, we only present an informal discussion on its proof in the following remark.

Remark. By a change of variables $x = x_0 + \sqrt{\varepsilon} z$, we obtain

$$\int_a^b \mathrm{e}^{-\frac{f(x)}{\varepsilon}} dx = \sqrt{\varepsilon} \int_{-\frac{x_0 - a}{\sqrt{\varepsilon}}}^{\frac{b - x_0}{\sqrt{\varepsilon}}} \mathrm{e}^{-\frac{1}{\varepsilon} f(x_0 + \sqrt{\varepsilon} z)} dz \,.$$

Notice that x_0 is a minimal point implies $f'(x_0) = 0$. Therefore, Taylor's expansion gives

$$\frac{1}{\varepsilon} f(x_0 + \sqrt{\varepsilon} z) = \frac{f(x_0)}{\varepsilon} + \frac{f''(x_0)}{2} z^2 + \mathcal{O}(\sqrt{\varepsilon} z^3) \,,$$

and

$$\begin{aligned}
\int_a^b \mathrm{e}^{-\frac{f(x)}{\varepsilon}} dx =& \sqrt{\varepsilon} \int_{-\frac{x_0 - a}{\sqrt{\varepsilon}}}^{\frac{b - x_0}{\sqrt{\varepsilon}}} \mathrm{e}^{-\frac{1}{\varepsilon} f(x_0 + \sqrt{\varepsilon} z)} dz \\
=& \sqrt{\varepsilon} \mathrm{e}^{-\frac{f(x_0)}{\varepsilon}} \int_{-\frac{x_0 - a}{\sqrt{\varepsilon}}}^{\frac{b - x_0}{\sqrt{\varepsilon}}} \mathrm{e}^{-\frac{f''(x_0)}{2} z^2 + \mathcal{O}(\sqrt{\varepsilon} z^3)} dz \\
=& \sqrt{\varepsilon} \mathrm{e}^{-\frac{f(x_0)}{\varepsilon}} \left(\sqrt{\frac{2\pi}{f''(x_0)}} + o(1) \right) .
\end{aligned}$$

Bibliography

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