FU course: Mathematical Strategies for Complex Stochastic Dynamics

April 30, 2025

Last week

ODEs:

$$\frac{dx(t)}{dt} = f(x(t)), \quad x(0) = x. \tag{1}$$

- C^1 smooth, deterministic, operator $(\mathcal{L}g)(x) := f(x) \cdot \nabla g(x)$.
  $\frac{d(g(x(t))}{dt} = \nabla g \cdot \frac{dx(t))}{dt} = \mathcal{L}g(x(t))$.
- gradient system: $\frac{dx(t)}{dt} = -\nabla V(x(t))$
  $\frac{dV(x(t))}{dt} = -|\nabla V|^2 \Longrightarrow \quad V(x(t))$ is non-decreasing $\Longrightarrow$ converge to local minima
- Hamiltonian system:

$$\frac{dq(t)}{dt} = \nabla_p H, \quad \frac{dp(t)}{dt} = -\nabla_q H. \tag{2}$$

  $\frac{dH(q(t),p(t))}{dt} = 0 \Longrightarrow H(q(t), p(t))$ is conserved.
- numerical scheme: Euler, Runge-Kutta, semi-implicit Euler...

Brownian motion

Brownian motions

Definition

A Brownian motion $(B(s))_{s \geq 0}$ in $\mathbb{R}^d$ has the following properties:

1. $B(0) = 0$.
2. $B(t)$ is almost surely continuous.
3. $B(t)$ has independent increments, i.e. $B(t) - B(s)$ is independent of $B(s)$.
4. $B(t) - B(s) \sim \mathcal{N}(0, (t - s)I_d)$.

Figure: Brownian motion

Brownian motions

The property $B(t) - B(s) \sim \mathcal{N}(0, (t-s)I_d)$ implies that:

1. The transition density is

$$p(x, t|y, s) = (2\pi(t-s))^{-\frac{d}{2}} \mathrm{e}^{-\frac{|x-y|^2}{2(t-s)}} . \tag{3}$$

2. In particular, the probability density at time t is

$$p(x, t) = (2\pi t)^{-\frac{d}{2}} \mathrm{e}^{-\frac{|x|^2}{2t}} . \tag{4}$$

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Proposition

The probability density p satisfies

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta p \,. \tag{5}$$

Ito integration

Ito integration in 1D

- Let $T > 0$, B_s be a Brownian motion in $\mathbb{R}$, and $f : [0, T] \to \mathbb{R}$ be either deterministic or random. We want to define $\int_0^T f(t) dB_t$.
- For any integer N, consider a uniform partition of $[0, T]$:

$$0 = t_0 < t_1 < \cdots < t_N = T, \text{ where } \quad t_n = nh, \quad h = \frac{T}{N}. \tag{6}$$

Recall that

$$\int_0^T f(t) dt = \lim_{N \to +\infty} \sum_{n=0}^{N-1} f(t_n)(t_{n+1} - t_n). \tag{7}$$

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Definition (Ito integration)

$$\int_0^T f(t) dB_t = \lim_{N \to +\infty} \sum_{n=0}^{N-1} f(t_n) \Delta_n, \text{ where } \Delta_n = B_{t_{n+1}} - B_{t_n}. \tag{8}$$

Ito integration in 1D

- Brownian increments:

$$\Delta_n = B_{t_{n+1}} - B_{t_n} \sim \mathcal{N}(0, hI_d) \quad \Longrightarrow \quad \mathbb{E}(\Delta_n | B_{t_n}) = 0, \quad \mathbb{E}(\Delta_n^2 | B_{t_n}) = h.$$

- We have

$$\mathbb{E}\Big(\int_0^T f(t) dB_t\Big) = 0, \quad \mathbb{E}\Big(\int_0^T f(t) dB_t\Big)^2 = \int_0^T f^2(t) dt\,. \tag{9}$$

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An infinitesimal version of the second identity is

$$(dB_t)^2 = dt\,, \text{ or } dB_t dB_s = \delta(t-s)dt\,. \tag{10}$$

Stochastic differential equations (SDEs)

SDEs

- X_t solves the SDE

$$dX_t = f(X_t)\,dt + \sigma(X_t)dB_t\,, \quad t > 0 \tag{11}$$

if and only if

$$X_t = \int_0^t f(X_s)\,ds + \int_0^t \sigma(X_s)dB_s\,, \quad t > 0 \tag{12}$$

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- Let $\eta_n \sim \mathcal{N}(0,1)$, independent. $\sqrt{h}\eta_n$ and Δ_n have the same distribution.
  Euler-Maruyama scheme:

$$x_{n+1} = x_n + f(x_n)h + \sqrt{h}\sigma(x_n)\eta_n\,, \quad n = 0, 1, 2, \cdots. \tag{13}$$

  SDE (11) can be thought as the limit of (13) as $h \to 0$.

Ito's lemma

Motivation:

- Smooth function $g : \mathbb{R} \to \mathbb{R}$.
- For ODEs, we have defined $(\mathcal{L}g) := f \cdot \nabla g$, and $\frac{d(g(x(t))}{dt} = \mathcal{L}g(x(t))$.
- Ito's lemma allows us to compute $dg(X_t)$ for SDEs.

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Lemma (Ito's lemma)

Assume that $dX_t = f(X_t)\,dt + \sigma(X_t)dB_t\,,\, t > 0$. *Then, we have*

$$dg(X_t) = \left[g'(X_t)f(X_t) + \frac{1}{2}g''(X_t)\sigma(X_t)^2\right]dt + \sigma(X_t)g'(X_t)dB_t\,. \tag{14}$$

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Infinitesimal generator: $\mathcal{L}g = fg' + \frac{1}{2}\sigma^2 g''$. Lemma 3 becomes

$$dg(X_t) = \mathcal{L}g(X_t)dt + \sigma(X_t)g'(X_t)dB_t\,.$$

Intuition of Ito's lemma

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Concretely, using $dX_t = f(X_t)\,dt + \sigma(X_t)dB_t$, we have

$$
\begin{aligned}
dg(X_t) =& g'(X_t)dX_t + \frac{1}{2}g''(X_t)(dX_t)^2 \\
=& g'(X_t)\Big(f(X_t)dt + \sigma(X_t)dB_t\Big) + \frac{1}{2}g''(X_t)\sigma(X_t)^2(dB_t)^2 \\
=& \Big(g'(X_t)f(X_t) + \frac{1}{2}g''(X_t)\sigma(X_t)^2\Big)dt + g'(X_t)\sigma(X_t)dB_t \\
=& (\mathcal{L}g)(X_t)dt + g'(X_t)\sigma(X_t)dB_t
\end{aligned}
\tag{15}
$$

Application of Ito lemma

Ito lemma imples

$$g(X_t) = \int_0^t \mathcal{L}g(X_s)ds + \int_0^t g'(X_s)\sigma(X_s)dB_s\,. \tag{16}$$

Application of Ito lemma

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Proposition

For a smooth function $g : \mathbb{R} \to \mathbb{R}$. We have

$$\mathbb{E}\big(g(X_t)\big) = \int_0^t \mathbb{E}\big(\mathcal{L}g(X_s)\big)ds\,, \quad \text{or} \quad \frac{d}{dt}\mathbb{E}\big(g(X_t)\big) = \mathbb{E}\big(\mathcal{L}g(X_t)\big)\,. \tag{17}$$

Semigroup and Fokker-Planck equation

Semigroup

Definition

Define

$$(T_t g)(x) = \mathbb{E}\big[g(X(t))|X(0) = x\big], \quad \forall t \geq 0\,, \forall g\,. \tag{18}$$

Proposition

We have $T_0 = id$*, and* $T_{t+s} = T_t \circ T_s$*, for* $t, s \geq 0$*. Moreoever,*

$$\frac{d}{dt} T_t g = \mathcal{L} T_t g\,. \tag{19}$$

Fokker-Planck equation

Recall

$$\mathcal{L}g = fg' + \frac{1}{2}\sigma(t)^2 g''.$$

Definition (Adjoint operator of $\mathcal{L}$)

Define $\mathcal{L}^\top$ such that

$$\int_{\mathbb{R}^d} \left[(\mathcal{L}^\top g)(x)\right] g_1(x) dx = \int_{\mathbb{R}^d} g(x) \left[\mathcal{L} g_1(x)\right] dx\,, \tag{20}$$

for any two smooth functions g and g_1.

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Lemma

$\mathcal{L}^\top g = -(fg)' + \frac{1}{2}(\sigma^2 g)'$.

Fokker-Planck equation

Assume $X_s = y$. Let $p(x, t)$ denote the probability density of X_t at time t, where $t \geq s$.

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Proposition

The density $p(x,t)$ satisfies the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \mathcal{L}^\top p \,, \quad t > s \,,$$
$$p(x,s) = \delta(x-y),$$

SDEs in $\mathbb{R}^d$

- Generator $\mathcal{L}$:

$$\mathcal{L}g = f \cdot \nabla g + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^\top)_{ij} \frac{\partial^2 g}{\partial x_i \partial x_j} \,.$$

- Adjoint $\mathcal{L}^\top$:

$$\int_{\mathbb{R}^d} \left[(\mathcal{L}^\top g)(x)\right] g_1(x) dx = \int_{\mathbb{R}^d} g(x) \left[\mathcal{L} g_1(x)\right] dx \,,$$

$$\Longrightarrow \quad \mathcal{L}^\top g = -\mathrm{div}(fg) + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 \left((\sigma\sigma^\top)_{ij} g\right)}{\partial x_i \partial x_j} \,.$$

Brownian motion

Example

Brownian motion $X_t = B_t$. The SDE is

$$dX_t = dB_t\,.$$

$f = 0$ and $\sigma = I_d$. The generator $\mathcal{L}$ and its adjoints are

$$\mathcal{L} = \mathcal{L}^\top = \frac{\Delta}{2}.$$

Therefore, Fokker-Planck equation reduces to the heat equation.